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BEZOUT THEOREM FOR NASH FUNCTIONS

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We define the complexity of Nash functions and give a Bezout theorem which uses this complexity. Then we obtain an upper bound for the sum of the Betti numbers of a Nash set.

1. Introduction

Let f be a polynomial map $f:\mathbf{R}^n\to\mathbf{R}^k$, $f=(f_1,\dots,f_k)$ such that for each $i=1,\dots,k$, $\deg f_i\le d$ and $Z_f=f^{-1}(0)$. Milnor [7] and Thom [12] give an upper bound for the sum of the Betti numbers of Z_f:

$$\sum_{i=0}^{n} b_i(Z_f)\le d(2d-1)^{n-1}. \tag{1}$$

In particular, this gives an upper bound for the number of connected components of Z_f. This result is used by Ben-Or [1] to produce lower bounds in algorithm complexity. Another application can also be found in [5] for the number of configurations and polytopes in $\mathbf{R}^d$.

Several works have been carried out to improve the bounds in (1). These are mainly centered on 3 ways:

(1) Smith's theory and topology of complex projective complete intersection are used to obtain a better bound in equation (1) [3]. More precisely, note that the bound in (1) does not depend on k. The importance of the parts played by k and n is distinguished. Then, in some cases, a better bound than the previous one is obtained.

(2) The bound in (1) is expressed according to other invariants of f like number of monomials, additive complexity [6, 10, 11].

(3) To obtain a similar relation to (1) in a 'larger class of functions' than the polynomials: Nash functions, Liouville functions, Pfaff functions, ... Already, we note that one of the main difficulties in this case is the definition of a notion, named complexity, which will replace the degree of a polynomial. The minimal properties that this complexity should satisfy are mentioned in [2].

Our aim is located in this third way, using as a frame the Nash functions: C^∞ semi-algebraic functions, defined in an open semi-algebraic U of R^n, where R

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denotes a real closed field. In the case of $R=\mathbf{R}$, Nash functions are equivalent to analytic functions which satisfy a polynomial equation $P(x,f(x))=0$, where P is a polynomial in $R[X,Y]$ which does not vanish identically [4]. Then we introduce the notion of complexity of Nash function as being the total degree of its minimal polynomial. It is the obvious generalization of the degree of a polynomial.

In [9], we show that there exists a semi-algebraic set which parametrizes the set of Nash functions having complexity smaller or equal to an integer d. This leads to the existence of uniform bounds for quite a lot of problems: Łojasiewicz's inequality, Positivstellensatz, approximation theorem etc. In particular, we prove the finiteness of topological types of Nash sets for a given complexity.

In this paper, we prove a Bezout theorem for Nash function (Section 3). Then, following Milnor's ideas, we obtain an upper bound for the sum of the Betti numbers of a Nash set V, depending on the complexity of the different functions which define V.

2. Nash functions. Complexity

Let U an open semi-algebraic of R^n, where R denotes a real closed field.

A function $f:U\to R$ is called semi-algebraic if its graph is a semi-algebraic set of R^{n+1}.

A function $f:U\to R$ is a Nash function if it is semi-algebraic and C^∞. The following lemma is a well-known result about Nash functions [4]:

2.1. Lemma. *There exists a polynomial $P\in R[X,Y]$, which does not vanish identically, such that for all $x\in U: P(x,f(x))=0$.* □

2.2. Definition. Let $f:U\to R$ be a Nash function.

The complexity of f, noted $c(f)$, is the minimum of the total degree of polynomials $P\in R[X,Y]$, which do not vanish identically and satisfy for all $x\in U$: $P(x,f(x))=0$, i.e.

$$c(f)=\mathrm{Min}\{\deg P \mid P\in R[X,Y],\ P(x,f(x))=0 \text{ for all } x\in U\}.$$

This definition is then an obvious generalization of a polynomial degree.

Let us recall some results about complexity of a sum, a product, and a derivative.

2.3. Proposition. *Let f and g be Nash functions. Then we have*:

(1) $c(f+g)\le c(f)\cdot c(g)$.

(2) $c(f\cdot g)\le 2c(f)\cdot c(g)$.

(3) $c(f^2)\le 2c(f)$.

(4) $$c(f_1^2+\cdots+f_p^2)\leq 2^p\prod_{i=1}^{i=p} c(f_i).$$

(5) $$c\left(\frac{\partial f}{\partial x_i}\right)\leq c(f)^2.$$

Proof. (1, 2, 5) See [9].

(3) Let $P(x,y)=a_n(x)y^n+\cdots+a_0(x)$ be the minimal polynomial of f. We split $P(x,f(x))$ into 2 parts:

$$P(x,f(x)) = \sum_{i\,\text{even}} a_i(x)f^i+\sum_{i\,\text{odd}} a_i(x)f^i = 0.$$

Then, we obtain

$$\left(\sum_{i\,\text{even}} a_i(x)f^i\right) = -f\left(\sum_{i\,\text{odd}} a_i(x)f^{i-1}\right)$$

and

$$\left(\sum_{i\,\text{even}} a_i(x)f^i\right)^2 = f^2\left(\sum_{i\,\text{odd}} a_i(x)f^{i-1}\right)^2.$$

So

$$Q(x,y) = \left(\sum_{i\,\text{even}} a_i(x)y^i\right)^2 - y^2\left(\sum_{i\,\text{odd}} a_i(x)y^{i-1}\right)^2$$

is a polynomial of degree less than or equal to $2c(f)$ and $Q(x,f^2(x))=0$.

(4) Easy consequence of (1) and (3). □

3. Bezout theorem

3.1. Theorem. *Let U be a connected open semi-algebraic of R^n and $f_1,\dots,f_n$ Nash functions of complexity $c_1,\dots,c_n$ defined in U.*

Then the number of non-degenerated solutions of the system

(S) $$\begin{cases} f_1(x)=0,\\ \quad\vdots \\ f_n(x)=0\end{cases}$$

is finite and less than or equal to $\prod_{i=1}^n c_i$.

Proof. A point $x_0=(x_1^0,\dots,x_n^0)\in R^n$ is a non-degenerated solution of (S) if and only if the jacobian $J(x)$ of $f_1(x),\dots,f_n(x)$ is not zero at x_0.

$$J(x_0)=\begin{vmatrix} \dfrac{\partial f_1}{\partial x_1}(x_0) & \cdots & \dfrac{\partial f_1}{\partial x_n}(x_0)\\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_n}{\partial x_1}(x_0) & \cdots & \dfrac{\partial f_n}{\partial x_n}(x_0)\end{vmatrix}.$$

Let $F_1(x,y),\dots,F_n(x,y)$ be polynomials of minimal total degree such that $F_i(x,f_i(x))=0$.

Consider the system

$$(\Sigma)\qquad \begin{cases} F_1(x,0)=0, \\ \quad\vdots \\ F_n(x,0)=0. \end{cases}$$

Any non-degenerated solution x_0 of (S) is a solution of (Σ). Then we can bound, by using the Bezout theorem for polynomials, the number of non-degenerated solutions of (S) by those of (Σ).

However, x_0 may be a degenerated solution of (Σ) and is not taken into account when we apply the Bezout Theorem to this system. Therefore we must proceed differently.

Let $\varepsilon=(\varepsilon_1,\dots,\varepsilon_n)\in\mathbf{R}^n$. Consider the system (S_ε) obtained by making a 'small perturbation to S'.

$$\begin{cases} f_1(x)=\varepsilon_1, \\ \quad\vdots \\ f_n(x)=\varepsilon_n. \end{cases}$$

Let x_0 be a non-degenerated solution of (S). By the local inverse theorem, there exists a non-degenerated solution $x=\varphi(\varepsilon)$, near x_0 for ε close to 0.

We claim that we can choose $\varepsilon=(\varepsilon_1,\dots,\varepsilon_n)$ such that $\varphi(\varepsilon)$ is also a non-degenerated solution of the system (Σ_ε):

$$\begin{cases} F_1(\varphi(\varepsilon),\varepsilon_1)=0, \\ \quad\vdots \\ F_n(\varphi(\varepsilon),\varepsilon_n)=0. \end{cases}$$

Indeed, we have

$$\frac{\partial F_i}{\partial x_j}(\varphi(\varepsilon),\varepsilon_i) = -\frac{\partial F_i}{\partial y}(\varphi(\varepsilon),\varepsilon_i)\frac{\partial f_i}{\partial x_j}(\varphi(\varepsilon)).$$

So,

$$J(F(\varphi(\varepsilon))) = \begin{vmatrix} \frac{\partial F_1}{\partial x_1}(\varphi(\varepsilon),\varepsilon_1) & \dots & \frac{\partial F_1}{\partial x_n}(\varphi(\varepsilon),\varepsilon_1) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1}(\varphi(\varepsilon),\varepsilon_n) & \dots & \frac{\partial F_n}{\partial x_n}(\varphi(\varepsilon),\varepsilon_n) \end{vmatrix}$$

$$= (-1)^n J(\varphi(\varepsilon)) \prod_{i=1}^{n} \frac{\partial F_i}{\partial y}(\varphi(\varepsilon),\varepsilon_i).$$

Since $\varphi(\varepsilon)$ is a non-degenerated solution of (S_ε), also $J(\varphi(\varepsilon))\neq 0$. Then it is enough to prove that the set of germs at 0 of ε such that

$$\prod_{i=1}^{n} \frac{\partial F_i}{\partial y}(\varphi(\varepsilon), \varepsilon_i) = 0$$

is of dimension less than or equal to n.

If not, we may assume that $\partial F_1/\partial y(\varphi(\varepsilon), \varepsilon_1) = 0$ in a neighbourhood W of 0.

$$\frac{\partial F_1}{\partial y}(x, f_1(x)) = \frac{\partial F_1}{\partial y}(\varphi(\varepsilon), \varepsilon_1)$$

will vanish identically in the neighbourhood $U \cap f^{-1}(W)$ of x_0. Since F_1 is the minimal polynomial of f_1, this gives a contradiction.

Then the number of non-degenerated solutions of (S) is bounded by the number of non-degenerated solutions of (Σ_ε) which is less than or equal to $\prod_{i=1}^{n} c_i$ by the Bezout theorem for polynomials. □

4. Bounds for the sum of the Betti numbers of a Nash set

4.1. Definition. A Nash set V in $\mathbf{R}^n$ is a semi-algebraic set which can be represented as

$$V = \{x \in \mathbf{R}^n \mid f_1(x) = \cdots = f_p(x) = 0\}$$

where f_i denotes a Nash function.

Let V be a Nash set. We denote by $H_i(V)$ the ith homology group of V with coefficients in $\mathbf{Z}/2\mathbf{Z}$. $H_i(V)$ is a $\mathbf{Z}/2\mathbf{Z}$-vector space; its dimension, denoted by $b_i(V)$, is called the ith Betti number of V. In particular, $b_0(V)$ is the number of connected components of V. Every $b_i(V)$ is finite and is null if $i \geq \dim(V)$. Then, the sum of the Betti numbers of V is always finite.

Let us recall that a function $g: V \to \mathbf{R}$ is a Morse function if g has only non-degenerate critical points.

On the other hand, according to Morse theory [8], if $g: V \to \mathbf{R}$ is a Morse function with V compact and non-singular, then the sum of the Betti numbers of V is less than or equal to the number of critical points of g.

4.2. Theorem. *Let V be a Nash set, compact and non-singular, defined by $f = 0$, where f denotes a Nash function of complexity $\leq d$. Then*

$$\sum (b_i(V)) \leq d^{2n-1}.$$

Proof. We follow Milnor's proof [7] step by step for introducing some control and explicit bounds. Let $\eta: V \to S^{n-1}$ be the function which assigns to each point $x \in V$ the unit normal vector. The set of critical values of η has dimension less than $(n-1)$. Then, there exist 2 points of S^{n-1} which are not critical values of η. Up to a rotation, we may assume that these points are $(0, \ldots, 1)$ and $(0, \ldots, -1)$. Remark that a rotation affects only the x_i and does not change the complexity. Let $h: V \to \mathbf{R}$ be the 'height function': $h(x_1, \ldots, x_n) = x_n$. Let us show that h is a Morse function. Let y be a critical point of h. We can take near y local coordinates: $x_1 = u_1, \ldots, x_{n-1} = u_{n-1}, x_n = h(u_1, \ldots, u_{n-1})$. We can compute that

$$\frac{\partial \eta_j}{\partial u_i}(y) = \pm \frac{\partial^2 h}{\partial u_i \partial u_j}(y).$$

The matrix $(\partial^2 h/\partial u_i \partial u_j(y))$ is non-singular; this means that h is a Morse function.

It follows, by Morse theory, that the sum of the Betti numbers of V is less than or equal to the number of critical points of h. They are the solutions of the system

$$\text{(S)} \qquad \begin{cases} \dfrac{\partial f}{\partial x_1} = 0, \\ \quad \vdots \\ \dfrac{\partial f}{\partial x_{n-1}} = 0, \\ f = 0. \end{cases}$$

Since h is a Morse function, y is a non-degenerated solution of (S). Hence, we can apply Bezout theorem to the system (S). Since each $\partial f/\partial x_i$ is a Nash function of complexity less than or equal to $c(f)^2 = d^2$, the theorem follows immediately. □

Now we want to remove the hypothesis that V is compact and non-singular.

4.3. Theorem. *Let V be a Nash set defined by $f_1(x) = \cdots = f_p(x) = 0$ where f_i denotes a Nash function of complexity less than or equal to d.*

Then the sum of the Betti numbers of V is less than or equal to $\frac{1}{2}(2^{p+1}d^p)^{2n-1}$.

Proof. For $R \geq 0$ sufficiently large, the inclusion $\overline{B(0,R)} \cap V \to V$ is a deformation retract. So, it is enough to bound $\sum b_i(\overline{B(0,R)} \cap V)$. For a given $\varepsilon \geq 0$, let F_ε be the Nash function defined by

$$F_\varepsilon(x) = f_1^2(x) + \cdots + f_p^2(x) + \varepsilon^2 \|x\|^2 - R^2.$$

$F_\varepsilon(x)$ has a complexity less than or equal to $2^{p+1}d^p$ ($\|x\|^2$ is of complexity 2).

Let $K_\varepsilon = \{x \in \mathbf{R}^n \mid F_\varepsilon(x) \leq 0\}$. K_ε is a compact set since it is contained in the disk $B(0, R/\varepsilon)$.

On the other hand, Sard's theorem gives us a real $a \geq 0$ such that for $\varepsilon \in]0, a[$, the boundary $\partial K_\varepsilon = \{x \in \mathbf{R}^n \mid F_\varepsilon(x) = 0\}$ of K_ε is non-singular. Then we can apply the above theorem to ∂K_ε:

$$\sum b_i(\partial K_\varepsilon) \leq (2^{p+1}d^p)^{2n-1}.$$

Now applying Alexander duality, it follows that

$$\sum b_i(K_\varepsilon) \le \frac{1}{2} \sum b_i(\partial K_\varepsilon) \le \frac{1}{2}(2^{p+1}d^p)^{2n-1}.$$

Since

$$\overline{B(0,R)} \cap V = \bigcap_{\varepsilon \in]0,a[} K_\varepsilon$$

and the fact that these sets can be triangulated, we have

$$H_i(\overline{B(0,R)} \cap V) = \varprojlim H_i(K_\varepsilon).$$

So,

$$\sum b_i(V) = \sum b_i(\overline{B(0,R)} \cap V) = \sum b_i(K_\varepsilon) \le \frac{1}{2}(2^{p+1}d^p)^{2n-1}. \quad \square$$

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